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SOME EXAMPLE APPLICATIONS OF COMPLEX NUMBERS

ДЕЯКІ ПРИКЛАДИ ЗАСТОСУВАННЯ КОМПЛЕКСНИХ ЧИСЕЛ

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Abstract. *The article is devoted to the application of complex numbers. The article explains the reasons for extending the set of real numbers to the set of complex numbers and describes their typical features. Although created in a purely theoretical mathematical way, complex numbers gradually found wide application in various fields. This article demonstrates the advantages of applying the theory of complex variables in some scientific and technical fields.*

Key words: *Complex numbers, forms a complex number, the theory of functions of a complex variable.*

Introduction.

Complex numbers are widely used across different branches of mathematics as well as among other sciences, including the elasticity theory, electrical engineering, fluid dynamics, aerodynamics, etc. Learning complex numbers expands the usual conception of numbers and number operations. As a result, it benefits abstract thinking development for students from the technical field.

Main part.

Introduction to the theory of functions of a complex variable is advisable to start with an overview of already known sets of numbers (N, Z, Q, R). Then the introduction can be continued with a cause analysis of the gradual extension of those sets [1].

I. In the process of counting, a set of natural numbers N arose, on which only the operations of addition and multiplication are closed. To perform the operation of subtraction of natural numbers, the set N is expanded, the number zero and negative integers are entered – this is the set of integers Z, on which the operation of division is not closed.

II. To perform the operation of dividing integers, the set of integers Z is expanded by introducing rational numbers Q (division by zero does not make sense).

III. There are incommensurable segments: if the length of one of such segments is taken as a unit of measurement, then the other of them, measured by this unit, is not expressed by a rational number. For example, the diagonal of a square and its side, taken as a unit of length, are incommensurable. Measuring incommensurable quantities, calculating such operations as root extraction, calculating logarithms, solving algebraic equations leads to further expansion of numbers, introduction of irrational numbers. In analysis, an irrational number is a non-periodic infinite decimal fraction.

The sets of rational and irrational numbers form the set of real numbers R.

According to the authors, the following characteristic features of real numbers should be highlighted:

- real numbers are one-dimensional;
- each number is represented by a point on the numerical axis;
- model only one property – quantity;
- assume comparison operations; subject to regulation;
- the square of any number is a non-negative value.

IV. The expansion of the set of real numbers to the set of complex numbers was due to the internal logic of the development of mathematics, namely the algebraic need to extract the square root of a negative number $\sqrt{-a},\ (a>0)$.

Complex numbers, as numbers of a new nature, were discovered in the XVIth century by the Italian scientist D. Cardano while searching for formulas for the roots of a cubic equation. Later, during the XVIIth - XIXth centuries, problems of complex numbers were dealt with by R. Bombelli (Italy), R. Descartes (France), L. Euler (Russia), K. Wessel (Denmark), R. Argand (France). The very term "complex numbers" was introduced by KF Gauss in 1831.

A complex number is a type number

$$z=a+bi, \tag{1}$$

where a and b are real numbers;

i – is the so-called imaginary unit determined from the condition $i=\sqrt{-1},\ (i^2=-1)$.

The numbers a and b are called the real and imaginary parts of the complex number z, respectively, and are denoted by $a=\operatorname{Re} z;\ b=\operatorname{Im} z$. If a complex number has a real component $a=0$, then the number $z=bi$ is called purely imaginary; if $b=0$, then the complex number $z=a$ is called pure real. Thus, real numbers are a subset of the set of complex numbers C.

The sets of numbers extension can be illustrated by the Euler diagram (Figure 1).

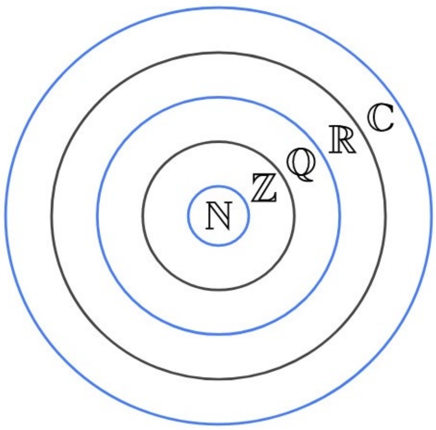

Figure 1 – Sets of numbers

Source: [2]

A complex number $z=a+bi$ is represented on the xOy plane by a point K with coordinates $(a;\ b)$ or by a vector $\overrightarrow{OK}$ whose length is $r=|OK|$ (the modulus $|z|$ of the complex number z), where $|z|=r=\sqrt{a^2+b^2}$, and the angle φ between the positive direction of the Ox axis and the vector $\overrightarrow{OK}$ (the argument of the complex number z); at the same time, due to the ambiguity of the definition of the angle with an accuracy of up to $2\pi n$ $\left(n \in Z\right)$, the argument is usually chosen within the limits $-\pi < \varphi \le \pi$ (the so-called main value of the argument) and is denoted by $\arg z$. It is obvious that

$a = r\cos\varphi,\ \ b = r\sin\varphi$, therefore from (1):

$$z = r\cos\varphi + i\, r\sin\varphi = r(\cos\varphi + i\sin\varphi). \qquad (2)$$

After Euler's derivation of the formula in 1748

$$e^{i\varphi} = \cos\varphi + i\sin\varphi, \qquad (3)$$

which connected the exponent function with trigonometric ones, the entry of the complex number (2) implies reformatting:

$$z = r\cos\varphi + i\, r\sin\varphi = re^{i\varphi}. \qquad (4)$$

Thus, complex numbers are presented in different forms of writing (table 1), which expands the possibilities of their practical application beyond the algebraic problem of obtaining the square root of a negative number.

Table 1 – forms of recording a complex number

Writing a complex number	Whole complex number	Purely imaginary	Purely valid
Algebraic form (1)	$z = a + bi$ $(z = \operatorname{Re} z + i \cdot \operatorname{Im} z)$	$z = bi$	$z = a$
Organized pair of numbers	$z = (a;\ b)$	$z = (0;\ b)$	$z = (a;\ 0)$
Trigonometric form (2)	$z = r(\cos\varphi + i\sin\varphi)$	$z = \lvert b\rvert i\sin\varphi$	$z = \lvert a\rvert\cos\varphi$
Exponential form (4)	$z = re^{i\varphi}$	$z = \lvert b\rvert e^{i\varphi}$	$z = \lvert a\rvert e^{i\varphi}$

Author's development

Let us note the features of complex numbers that are different from real numbers:

- complex numbers have two components, that is, they are two-dimensional;
- each number is represented by a point on the coordinate plane;
- model vector quantities on the plane;
- “more or less” operations are not defined on them;
- assume a root from any number.

Therefore, the introduction of complex numbers is a fundamentally new level of generalization of numbers with a new geometric interpretation of the concept of number.

We will demonstrate some advantages of applied application of complex numbers on examples from various scientific fields.

1. Definition of regions (in the topological sense) on the complex plane [3].

1.1. Since the equation of a circle of radius ρ with the center at a point z_0 has the form $|z - z_0| = \rho$, then inequalities

$$\rho_1 < \ |z - z_0|\ < \rho_2\ ,\ (0 < \rho_1 < \rho_2)$$

will set a ring with an inner radius ρ_1 and an outer radius ρ_2.

1.2. Since the ray emanating from the point z_0 at an angle φ has the equation $\arg(z - z_0) = \varphi$, then inequalities

$$\varphi_1 < \arg(z - z_0) < \varphi_2 , \quad \varphi_2 < \arg(z - z_0) < \varphi_1 + 2\pi , \quad (0 \le \varphi_1 < \varphi_2 < 2\pi)$$

will determine the angles formed by two rays starting at point z_0.

1.3. Strips bounded by a pair of parallel lines can be determined by the following inequalities:

$$c_1 < \operatorname{Re} z < c_2 , \ (0 < c_1 < c_2,\ c_1, c_2 \in R) \text{ – vertical stripes;}$$

$$c_1 < \operatorname{Im} z < c_2 , \ (0 < c_1 < c_2,\ c_1, c_2 \in R) \text{ – horizontal stripes.}$$

2. Calculation of trigonometric sums.

2.1. In problems of optics about the diffraction grating [4] sums of the form appear

$$S_n(t) = \sin \omega t + \sin(\omega t + \varphi) + ... + \sin(\omega t + n\varphi) . \tag{5}$$

Let's use the indicative form of recording complex numbers and formula (3):

$$\operatorname{Im}(e^{i\varphi}) = \sin\varphi ,$$

$$S_n(t) = \operatorname{Im}(e^{i\omega t}) + \operatorname{Im}(e^{i(\omega t+\varphi)}) + ... + \operatorname{Im}(e^{i(\omega t+n\varphi)}) =$$

$$= \operatorname{Im}(e^{i\omega t} + e^{i(\omega t+\varphi)} + ... + e^{i(\omega t+n\varphi)}).$$

According to the formula for the sum of members of a geometric progression:

$$e^{i\omega t} + e^{i(\omega t+\varphi)} + ... + e^{i(\omega t+n\varphi)} = \frac{e^{i(n+1)\varphi} - 1}{e^{i\varphi} - 1} e^{i\omega t} .$$

Therefore,

$$S_n(t) = \operatorname{Im}\left(\frac{e^{i(n+1)\varphi} - 1}{e^{i\varphi} - 1} e^{i\omega t}\right) . \tag{6}$$

2.2. Let's calculate, in particular, the amount.

$$A_n = \sum_{k=1}^{n} \sin k\varphi . \tag{7}$$

It is obvious that A_n is equal to the sum (5) at $t = 0$:

$$A_n = S_n(0).$$

Applying (3), (4), we isolate the imaginary part in (6) at $t = 0$:

$$\operatorname{Im}\left(\frac{e^{i(n+1)\varphi} - 1}{e^{i\varphi} - 1} e^{i\cdot 0}\right) = \operatorname{Im}\left(\frac{e^{\frac{i(n+1)\varphi}{2}}\left(e^{\frac{i(n+1)\varphi}{2}} - e^{\frac{-i(n+1)\varphi}{2}}\right)}{e^{\frac{i\varphi}{2}}\left(e^{\frac{i\varphi}{2}} - e^{\frac{-i\varphi}{2}}\right)}\right) = \operatorname{Im}(e^{\frac{in\varphi}{2}}) \cdot \frac{2i\sin\frac{(n+1)\varphi}{2}}{2i\sin\frac{\varphi}{2}} ;$$

where $A_n = \dfrac{\sin\frac{n\varphi}{2} \cdot \sin\frac{(n+1)\varphi}{2}}{\sin\frac{\varphi}{2}}$.

3. The theory of functions of a complex variable is used when finding solutions of linear differential equations with constant coefficients. Differential equations of this type often appear in mathematical modeling [11] in the theory of oscillations of a material point in the environment [6], the theory of automatic control [7], processes in electrical engineering [8], economic processes [9], etc.

Consider, for example, the equation of the dependence of the current strength $I(t)$ of a closed circuit with the specified coefficient of self-induction L, resistance R, electromotive force $E = E_0 \sin \omega t$:

$$\frac{dI}{dt} + \alpha I = \frac{E_0}{L} \sin \omega t, \ \alpha = \frac{R}{L}.$$

When solving this linear inhomogeneous differential equation by the Bernoulli method, it will be necessary to calculate the integral

$$B = \int e^{\alpha t} \sin \omega t dt.$$

Since $e^{\alpha t} \sin \omega t = \mathrm{Im}(e^{\alpha t} e^{i\omega t})$, then $B = \mathrm{Im} \int e^{(\alpha + i\omega)t} dt$, from which is obtained [10]:

$$B = \frac{e^{\alpha t}}{\omega^2 + \alpha^2}(-\omega \cos \omega t + \alpha \sin \omega t) + C.$$

4. Modeling of images when displaying plane points [2].

4.1. Parallel transfer, which translates point $z_1 = (a_1;\ b_1)$ to point $z_2 = (a_2;\ b_2)$:

$$z \to z + (z_2 - z_1) = z + ((a_1 + a_2) + i(b_1 + b_2)).$$

4.2. Rotation around the origin by an angle α:

$$z \to e^{\alpha t} \cdot z = (\cos \alpha + i \sin \alpha) \cdot z.$$

4.3. Axial symmetry:

$z \to \mathrm{Re}\, z - i \cdot \mathrm{Im}\, z = \overline{z}$ (symmetry relative to the axis *Ox*),

$z \to i \cdot \mathrm{Im}\, z - \mathrm{Re}\, z$ (symmetry relative to the axis *Oy*).

Conclusion.

Thus, according to the authors, when studying complex numbers, one should:

- ✓ find out the reasons for the expansion of numerical sets to the set of complex numbers;
- ✓ describe the possibilities of presenting a complex number in different forms of writing (algebraic, ordered by a pair of real numbers, trigonometric, exponential) and transition from one form to another;
- ✓ establish characteristic features of complex numbers;
- ✓ give specific examples of some applied applications in which one or another form of writing a complex number is used, in particular, the advantage of the index form of writing a complex number is demonstrated (short number writing, convenience in multiplication, division and exponentiation, the possibility of applying the Euler formula (3)).

Consequently, although created in a purely theoretical mathematical way, complex numbers gradually found wide application in various fields (aerodynamics, electromagnetic waves theory, geodesy, electrical engineering, signal processing, mathematical economic models, theory of relativity, etc).

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Анотація. *Дана стаття присвячена окремим прикладним застосуванням теорії функції комплексної змінної.*

У статті з'ясовані причини розширення числових множин: так операція віднімання привела до появи від'ємних чисел, операція ділення – раціональних чисел, добування кореня – ірраціональних чисел. Всі ці множини входять у множину комплексних чисел, в якій виконується операція добування кореня із будь-якого числа.

Комплексні числа завдяки таким принципово новим характеристикам поняття числа як двокомпонентність, геометрична інтерпретація точкою або вектором на площині, можливість переходу від однієї форми запису комплексного числа (алгебраїчної, впорядкованою парою дійсних чисел, тригонометричної, експоненціальної) до іншої, отримали широке застосування у різних науково технічних галузях, зокрема, в аеродинаміці, теорії електромагнітних хвиль, геодезії, електротехніці, теорії сигналів, математико-економічному моделюванню, теорії відносності та ін.

Ключові слова: *комплексні числа, форми запису комплексного числа, теорія функції комплексної змінної.*