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OPERATION OF REVERSING OF LOGICAL MATRICES

ОПЕРАЦІЯ ОБЕРТАННЯ ЛОГІЧНИХ МАТРИЦЬ

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Abstract. *In classical linear algebra the machine of matrices is widely used. But the classic linear algebra deals with continuous objects. Logical algebra, built by analogy with the classical linear algebra, builds the same models using discrete objects that have logical structure and obey the relevant laws. This leads to a significant difference in the functioning of the constructed models. This article is devoted to matrices, as elements for which the elementary logical elements are taken Boolean constants or finite predicates of arbitrary arity. In the work investigated the features of the operation of reversing of such matrices. All the results obtained are illustrated with clear examples.*

Keywords: *finite predicate, Boolean matrix, predicative matrix, disjunction, conjunction, inversion, orthographic logical matrix, logical scalar, reversing of logical matrix.*

Introductions.

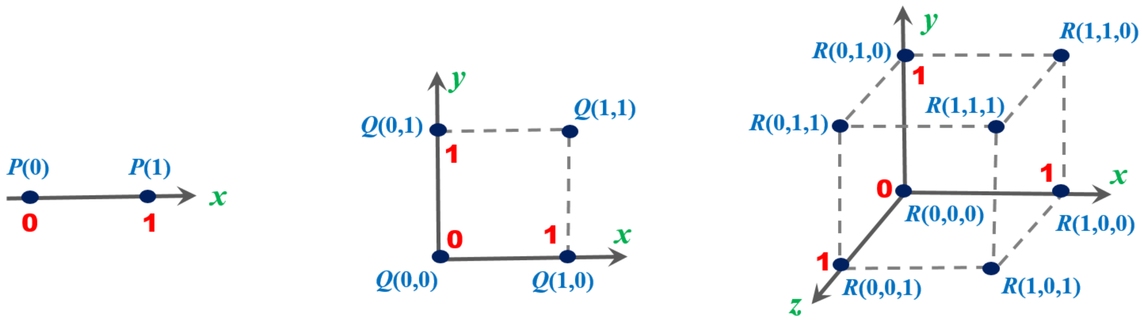
Established ideas about mathematical logic as a science that studies the laws of thinking using the apparatus of mathematics, mainly for the needs of mathematics itself, are becoming too narrow in modern conditions. With the expansion of the fields of application and the further development of mathematical logic, the view on it also changes. The objects of mathematical logic are any discrete finite systems, and its main task is the structural modeling of such systems. For the description of many logical processes and phenomena, for example, natural human language, an apparatus of equations similar to the apparatus used in mathematical analysis, but different from the latter in that it is intended to formalize not continuous, but discrete processes, would be best suited. Such a language is given by logical deductions, namely: deduction of statements and deduction of predicates. However, in order to be able to effectively solve these equations, it is necessary to bring these calculations to the level of an algebraic system. One of the most important algebraic models is the apparatus of matrices. This apparatus is the basis for the construction and further research of any vector spaces. At the same time, one of the most important operations on matrices is their rotation. For matrices defined over the field of logical scalars, this operation has certain features.

Main text.

There are two types of logical matrices: Boolean and predicate. A logical matrix is called a Boolean matrix if its elements are logical scalars from the field *G*={0, 1} [1]. That is, the elements of the Boolean matrix are zeros and ones. In turn, a logical matrix is called predicate if all its elements are taken from the same field of finite predicates of arbitrary arity.

Any function $t=f(x_1, x_2, \dots, x_n)$ of *n* letter arguments $x_1, x_2, \dots, x_n$ given on the set *A*, which accepts logical the value of *t*. Sometimes the finite predicate *f* is called *k*-element, emphasizing that its alphabet *A* consists of *k* letters [2].

The elements of predicate scalar fields, depending on the arity of the predicate, can be presented as follows graphical models:



a) single-place predicate ***b) double-place predicate*** ***c) triple-place predicate***

Figure 1 – Graphic representation of predicate logical scalars

Author's development

Thus, each element of the predicate logical matrix can be presented in the form of a hypercube of dimension *n* [3]. For example, in pic. 1c) considered the case of the triple-place predicate *R*(*x*, *y*, *z*), which is given over the alphabet *G*={0, 1} with *k*=2 letters. At the same time, each vertex of the hypercube corresponds to the value of the predicate at certain values of the arguments *x*, *y*, and *z*, which formed this vertex. The role of the unit element of the scalar field is played by the predicate, which is equal to one for all values of its arguments. Accordingly, the role of the null element is played by the predicate, which for all values of its arguments is equal to zero. Graphically, a unit element is represented by a hypercube with all vertices corresponding to ones, and a zero element by a hypercube with all vertices corresponding to zeros.

All operations on the elements of the logical scalar field are performed bit by bit. The category means the value of the considered predicate with one of the possible sets of arguments. Thus, binary operations (disjunction and conjunction) assume that their result will be an element, each bit of which corresponds to the value of the performed binary operation on the same-named bits of the predicates involved in the operation. The same-named categories mean the values of these predicates from the same sets of arguments. By the way, the unary negation operation is also performed bit by bit. These operations will be calculated according to the following rules:

$$(P_i \vee P_j)\,(x_1, x_2, \ldots, x_n) = P_i(x_1, x_2, \ldots, x_n) \vee P_j(x_1, x_2, \ldots, x_n), \quad (1)$$

$$(P_i \wedge P_j)\,(x_1, x_2, \ldots, x_n) = P_i(x_1, x_2, \ldots, x_n) \wedge P_j(x_1, x_2, \ldots, x_n), \quad (2)$$

$$\overline{P}_i(x_1, x_2, \ldots, x_n) = \overline{P_i(x_1, x_2, \ldots, x_n)}. \quad (3)$$

Graphic representation of these operations on the elements of the predicate matrices are shown in following pictures.

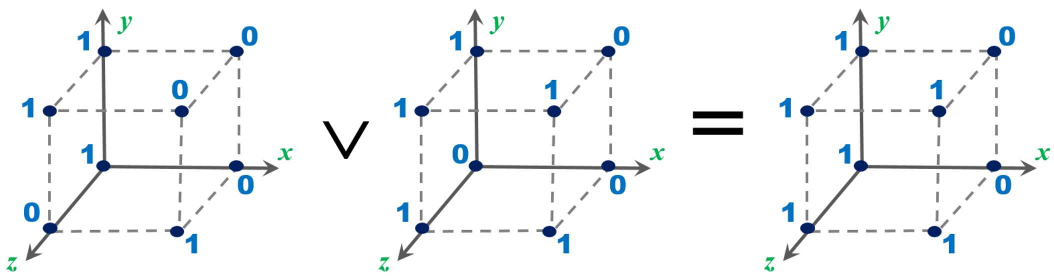


Figure 2 – Graphic representation of disjunction

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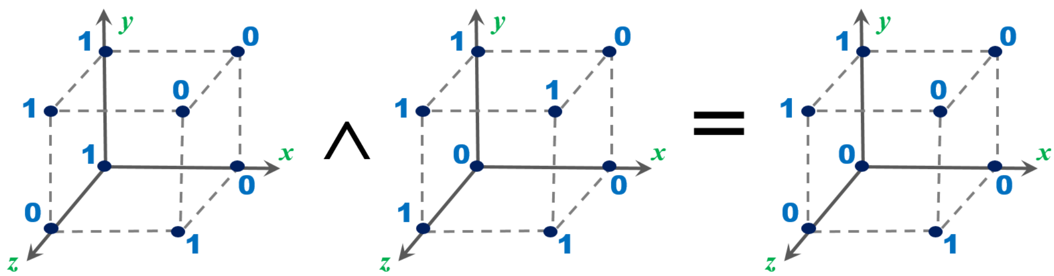

Figure 3 – Graphic representation of conjunction

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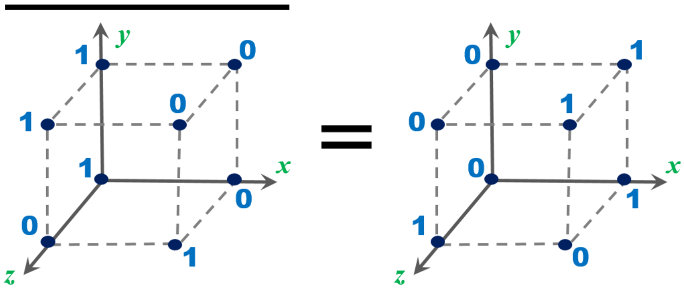

Figure 4 – Graphic representation of inversion

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Let us now consider the analytical representation of the predicate field of logical scalars. For example, the set of single-place predicates $P_i(x)$, i=0, 1, 2, 3, where $x \in \{0, 1\}$, can be considered as a field of logical scalars. This set is given by table 1.

Table 1 – The set of single-place predicates given on the alphabet G={0, 1}

x	P_0	P_1	P_2	P_3
0	0	0	1	1
1	0	1	0	1

A source: [4]

Next, single-digit predicates are represented by strings P=(P(0), P(1)). Then P_0=(0,0), P_1=(0,1), P_2=(1,0), P_3=(1,1). Let us denote this scalar field by ***P***. The set of logical scalars ***P*** given in this table is exhaustive.

As for the operation of reversing, in the case of predicate matrices, its definition is somewhat more complicated than for Boolean logic matrices [4]. If we consider Boolean matrices as a special case of predicate matrices (they are set over the field of zero-place predicates), then it can be argued that under this condition the following definitions and assertions apply to Boolean matrices as well. But for them, all this can be calculated more easily [1].

A square logical matrix A is called orthogonal if the disjunction of all elements of each of its rows and the disjunction of all elements of each of its columns are equal to the same unit. At the same time, the conjunction of any two elements in each of its rows and the conjunction of any two elements in each of its columns are identically equal to zero.

For example, the logical matrix A over the field $\boldsymbol{P}$ of single-place predicates

$$A = \begin{pmatrix} P_1 & P_0 & P_2 \\ P_2 & P_0 & P_1 \\ P_0 & P_3 & P_0 \end{pmatrix} = \begin{pmatrix} (0,1) & (0,0) & (1,0) \\ (1,0) & (0,0) & (0,1) \\ (0,0) & (1,1) & (0,0) \end{pmatrix}$$

is orthogonal, and the matrix

$$B = \begin{pmatrix} P_1 & P_3 & P_2 \\ P_0 & P_1 & P_0 \\ P_2 & P_0 & P_1 \end{pmatrix} = \begin{pmatrix} (0,1) & (1,1) & (1,0) \\ (0,0) & (0,1) & (0,0) \\ (1,0) & (0,0) & (0,1) \end{pmatrix} -$$

no, because for example $a_{12} \wedge a_{13} = P_3 \wedge P_2 = (1,1) \wedge (1,0) = (1,0) \neq 0$.

Theorem. For square logical matrices A and B over the field of logical scalars $G=\{0, 1\}$ or the field of finite predicates of arbitrary arity, the equality $AB=E$ holds, it is necessary and sufficient that A and B are orthogonal matrices and obey the condition $B=A^T$.

Proof. Let us assign to each rank of the elements of the scalar field, over which the matrices A and B are given, some index $v=1, ..., (k_1\ k_2\ \dots\ k_n)$, where n is the arity of predicates that are elements of the scalar field, and k_i, $i=\overline{1,n}$, the number of characters in the alphabet above which the argument xi of these predicates is given. Thus, matrices A and B decompose into $(k_1\ k_2\ \dots\ k_n)$ matrices over the Boolean set $G=\{0, 1\}$

$$A^v = \begin{pmatrix} a_{11}^v \ \dots \ a_{1s}^v \\ \dots\dots\dots\dots \\ a_{s1}^v \ \dots \ a_{ss}^v \end{pmatrix} \text{ and } B^v = \begin{pmatrix} b_{11}^v \ \dots \ b_{1s}^v \\ \dots\dots\dots\dots \\ b_{s1}^v \ \dots \ b_{ss}^v \end{pmatrix},$$

composed of the v-th ranks of elements of matrices A and B. Therefore, if the statement is valid for matrices over the Boolean field of scalars $G=\{0, 1\}$, then it is also true for matrices whose elements are predicates of arbitrary arity. Due to this, it suffices to prove the statement for the case of the scalar field $G=\{0, 1\}$ [2]. Let the size of the matrices A^v and B^v be $s \times s$. Let's choose an arbitrary integer t, $1 \leqslant t \leqslant s$. If the t-th row of the matrix A^v is zero, then the t-th row of the matrix $(AB)^v$ will also be zero. Therefore, in each row of the matrix A^v there is at least one unit, and this unit corresponds to some unit in the matrix B^v (let it be the element a^v_{tj}=1 to which b^v_{jt} corresponds). At $f \neq t$ ($1 \leqslant f \leqslant s$) we have a^v_{fj}=0, because otherwise $(AB)^v_{ft}=a^v_{fj}\, b^v_{jt}$=1, i.e. $(AB)^v \neq E$. Similarly, in matrix B^v, all elements of row j, except b^v_{jt}, are equal to zero. Thus, in each row of the matrix A^v there is at least one unit, and all these units are located in different columns. Therefore, the matrix A^v is orthogonal. Similarly, the matrix B^v is also orthogonal. The equality $B^v=(A^v)^T$ is now obvious (every element of a^v_{tj}=1 corresponds to b^v_{jt}=1). *The theorem is proved*.

Consider the above orthogonal matrix A as an example. According to the theorem just proved, its inverse will be its transposed matrix A^T:

$$A^T = \begin{pmatrix} P_1 & P_0 & P_2 \\ P_2 & P_0 & P_1 \\ P_0 & P_3 & P_0 \end{pmatrix}^T = \begin{pmatrix} (0,1) & (0,0) & (1,0) \\ (1,0) & (0,0) & (0,1) \\ (0,0) & (1,1) & (0,0) \end{pmatrix}^T =$$

$$= \begin{pmatrix} (0,1) & (1,0) & (0,0) \\ (0,0) & (0,0) & (1,1) \\ (1,0) & (0,1) & (0,0) \end{pmatrix} = \begin{pmatrix} P_1 & P_2 & P_0 \\ P_0 & P_0 & P_3 \\ P_2 & P_1 & P_0 \end{pmatrix}.$$

Obviously, the matrix A^T is also orthogonal. Let us now calculate the multiplication of the orthogonal matrix A by its transposed matrix:

$$A\,A^T = \begin{pmatrix} (0,1) & (0,0) & (1,0) \\ (1,0) & (0,0) & (0,1) \\ (0,0) & (1,1) & (0,0) \end{pmatrix} \cdot \begin{pmatrix} (0,1) & (1,0) & (0,0) \\ (0,0) & (0,0) & (1,1) \\ (1,0) & (0,1) & (0,0) \end{pmatrix} =$$

$$= \begin{pmatrix} (0,1)\wedge(0,1) \vee (0,0)\wedge(0,0) \vee (1,0)\wedge(1,0) & (0,1)\wedge(1,0) \vee (0,0)\wedge(0,0) \vee (0,1)\wedge(0,1) & (0,1)\wedge(0,0) \vee (0,0)\wedge(1,1) \vee (1,0)\wedge(0,0) \\ (1,0)\wedge(0,1) \vee (0,0)\wedge(0,0) \vee (0,1)\wedge(1,0) & (1,0)\wedge(1,0) \vee (0,0)\wedge(0,0) \vee (0,1)\wedge(0,1) & (1,0)\wedge(0,0) \vee (0,0)\wedge(1,1) \vee (0,1)\wedge(0,0) \\ (0,0)\wedge(0,1) \vee (1,1)\wedge(0,0) \vee (0,0)\wedge(1,0) & (0,0)\wedge(1,0) \vee (1,1)\wedge(0,0) \vee (0,0)\wedge(0,1) & (0,0)\wedge(0,0) \vee (1,1)\wedge(1,1) \vee (0,0)\wedge(0,0) \end{pmatrix} =$$

$$= \begin{pmatrix} (1,1) & (0,0) & (0,0) \\ (0,0) & (1,1) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{pmatrix} = E.$$

For the above matrix B, the operation of reversing cannot be performed, that is, there is no exist reverse matrix for this matrix. It follows that the proven rule for calculating the reverse matrix for the matrix B does not hold:

$$B\,B^T = \begin{pmatrix} P_1 & P_3 & P_2 \\ P_0 & P_1 & P_0 \\ P_2 & P_0 & P_1 \end{pmatrix} \cdot \begin{pmatrix} P_1 & P_3 & P_2 \\ P_0 & P_1 & P_0 \\ P_2 & P_0 & P_1 \end{pmatrix}^T =$$

$$= \begin{pmatrix} (0,1) & (1,1) & (1,0) \\ (0,0) & (0,1) & (0,0) \\ (1,0) & (0,0) & (0,1) \end{pmatrix} \cdot \begin{pmatrix} (0,1) & (1,1) & (1,0) \\ (0,0) & (0,1) & (0,0) \\ (1,0) & (0,0) & (0,1) \end{pmatrix}^T =$$

$$= \begin{pmatrix} (0,1) & (1,1) & (1,0) \\ (0,0) & (0,1) & (0,0) \\ (1,0) & (0,0) & (0,1) \end{pmatrix} \cdot \begin{pmatrix} (0,1) & (0,0) & (1,0) \\ (1,1) & (0,1) & (0,0) \\ (1,0) & (0,0) & (0,1) \end{pmatrix} =$$

$$= \begin{pmatrix} (0,1)\wedge(0,1) \vee (1,1)\wedge(1,1) \vee (1,0)\wedge(1,0) & (0,1)\wedge(0,0) \vee (1,1)\wedge(0,1) \vee (1,0)\wedge(0,0) & (0,1)\wedge(1,0) \vee (1,1)\wedge(0,0) \vee (1,0)\wedge(0,1) \\ (0,0)\wedge(0,1) \vee (0,1)\wedge(1,1) \vee (0,0)\wedge(1,0) & (0,0)\wedge(0,0) \vee (0,1)\wedge(0,1) \vee (0,0)\wedge(0,0) & (0,0)\wedge(1,0) \vee (0,1)\wedge(0,0) \vee (0,0)\wedge(0,1) \\ (1,0)\wedge(0,1) \vee (0,0)\wedge(1,1) \vee (0,1)\wedge(1,0) & (1,0)\wedge(0,0) \vee (0,0)\wedge(0,1) \vee (0,1)\wedge(0,0) & (1,0)\wedge(1,0) \vee (0,0)\wedge(0,0) \vee (0,1)\wedge(0,1) \end{pmatrix} =$$

$$= \begin{pmatrix} (1,1) & (0,1) & (0,0) \\ (0,1) & (0,1) & (0,0) \\ (0,0) & (0,0) & (1,1) \end{pmatrix} = \begin{pmatrix} P_3 & P_1 & P_0 \\ P_1 & P_1 & P_0 \\ P_0 & P_0 & P_3 \end{pmatrix} \neq E.$$

Conclusions.

To reversing logical matrices, both Boolean and predicate, it is impossible to apply the algorithms that exist for ordinary matrices. As it was proved, not every logical matrix can perform this operation. In order for the reverse matrix to exist, logical matrices, in addition to their dimensions, are subject to an additional restriction regarding the composition of their elements. In ordinary linear algebra, there are no such restrictions. This is one of the essential differences between the apparatus of logical algebra and the usual mathematical apparatus of linear vector spaces.

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Анотація. *В класичній лінійній алгебрі широко використовується апарат матриць. Але класична лінійна алгебра має справу з безперервними об'єктами. Логічна алгебра, побудована за аналогією з класичною лінійною алгеброю, створює ті ж самі моделі за допомогою дискретних об'єктів, що мають логічну структуру та підкоряються відповідним законам. Це призводить до суттєвих відмінностей у функціонуванні побудованих моделей. Дана стаття присвячена матрицям, в якості елементів для яких обираються елементарні логічні елементи, а саме булеві константи або скінченні предикати довільної арності. В роботі досліджено особливості операції обертання таких матриць. Всі отримані результати проілюстровано наочними прикладами.*

Ключові слова: *скінченний предикат, булева матриця, предикатна матриця, диз'юнкція, кон'юнкція, заперечення, ортогональна логічна матриця, логічний скаляр, обертання логічної матриці.*

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