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PROPAGATION OF GUIDED HARMONIC WAVES IN ANISOTROPIC LAMINATES

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Abstract. *In this study, a numerical method is presented to analyze the propagation characteristics of guided harmonic waves in multilayer anisotropic composite laminates. A generalized matrix of displacement and mechanical strain components for each local volume of the composite is obtained through theoretical analysis. A generalized case of anisotropic laminar composite is investigated in this paper. The displacement fields are approximately fitted by Legendre polynomials, and the system of linear equations is constructed using orthogonal projection. The eigenvalue/eigenvector solution is established to calculate the phase dispersion curves instead of solving the transcendental dispersion equations. The solution of the system of nonlinear equations for the components of mechanical shears can be represented for the case of plane harmonic waves as a superposition of symmetric and antisymmetric Lamb wave modes. The analysis was performed based on the wavelet transforms of the metric functions, primarily for the symmetric modes. It was found that for the case of using the Hermite matrix transforms, the eigenvalues for the symmetric and antisymmetric modes coincide. For each layer of the non-isotropic laminar composite, it was possible to represent the displacement components in exponential form.*

Key words: *laminar composites, wavelet transforms, Lamb waves, symmetrical modes, displacement matrix.*

Introduction.

Non-destructive testing, as well as structural health monitoring, determine the integrity and degradation of composite structures to ensure their operability. The working object of active diagnostics is the ultrasonic transient wave. In order to detect damage, localize and subsequently evaluate damage, understanding the wave propagation characteristics of composites is essential for the successful application of diagnostic methods.

The effects of wave propagation in composites are complex due to the nature of the component heterogeneity, inherent material anisotropy and multilayer construction. These features are the reason why the wave mode velocity is macroscopically dependent on the laminate layup, the direction of wave propagation, frequency and interface conditions.

The propagation of elastic waves in isotropic plate structures is characterized by repeated reflections on the upper and lower surfaces alternately [1]. As a result, the propagation of waves from their mutual interference is guided by the surfaces of the plates. The guided wave can be modeled by imposing surface boundary conditions on the equations of motion [2, 3]. The effects of wave propagation in composite structures are accompanied by the phenomenon of dispersion, i.e. the propagation velocity of a guided wave along the plate is a function of frequency or, equivalently, wavelength. In particular, guided waves propagating along the plane of an elastic plate with tension-free boundaries are called Lamb waves. Since guided waves remain confined within the structure, they can propagate over large distances, allowing a large area to be surveyed with only a limited number of sensors. This property makes them well suited for continuous monitoring techniques for ultrasonic testing of entire structures and their elements, which are used in various industrial fields.

Lamb waves polarized in the plane perpendicular to the plate, in the x-z plane, are called symmetric (or longitudinal, S) waves and antisymmetric (or flexural, A) waves, while those polarized in the plane of the plate (along the y-axis) are called shear horizontal (SH) waves [4]. SH waves can also be either symmetric or antisymmetric about the mid-plane. S and A waves are controlled by plane strain state (displacements u and w); while SH waves are controlled by anti-plane strain (displacement v only). Conventionally, Sn and An with index n = 0, 1, 2, 3... represent symmetric and antisymmetric Lamb wave modes, respectively; SHn with even and odd index n denote symmetric and antisymmetric SH waves, respectively. Wave interactions of waves propagating in multilayer composites depend on the properties of the components, geometry, direction of propagation, frequency, and conditions at the interface. For the case when the wavelengths are significantly larger than the dimensions of the composite components (fiber diameters and spacing), each plate can be considered as a sample made of an equivalent homogeneous orthotropic or transversely isotropic material.

Propagation of wave packets.

The displacement of plane harmonic waves can be described in general using 3-D elasticity. The initial stage of the analysis investigates the characteristics of Lamb waves in a single plate (monoclinic plate). In this case, a compact closed dispersion relation can be obtained by separating symmetric and antisymmetric modes using trigonometric functions through the plate thickness. Special cases are when the waves propagate along the symmetry axis of the material in such a way that mutual separation of S- and A waves as well as SH waves is considered. The final stage of the analysis generates a modified exponential form in the thickness direction to derive the dispersion relation for the composite laminate with special emphasis on symmetric laminates. The propagation of wave packets is considered in a Cartesian coordinate system with the z-axis perpendicular to the mid-plane of the composite laminate spanned by the x- and y-axes. The two outer surfaces of the laminate are defined by the coordinates $z = \pm h/2$. An arbitrary direction θ of the Lamb wave packet is defined counterclockwise with respect to the x-axis. In this case, a fixed layer of the composite laminate with an arbitrary orientation in the global coordinate system (x, y, z) is considered as a monoclinic material having x–y as a plane of symmetry.

The case where the global coordinate system (x, y, z) does not coincide with the main coordinate system of the material (x', y', z) of each layer, but forms an angle φ with the x-axis is considered separately. For such conditions, the stiffness matrix C_{ij} (i, j = 1, 2, 3, 6) in the system (x, y, z) can be obtained from the plate stiffness matrix C_{0ij} in the system (x_0, y_0, z) using the transformation matrix method. The composite material specimen is orthotropic or transversely isotropic with respect to the main axes of the material in (x_0 , y_0, z). The plate stiffness matrix C_{0ij} can be calculated from the plate material properties E_k, ν_{kl} and G_{kl} (k, l = 1, 2, 3).

The relationships between deformations and displacements are as follows

$$\varepsilon_x = u_x,\ \ \varepsilon_y = u_y,\ \ \varepsilon_z = u_z,\ \ \gamma_{yz} = \upsilon_z + w_y,$$
$$\gamma_{xz} = u_z + w_x,\ \ \gamma_{xy} = u_y + \upsilon_x, \tag{1}$$

where u is the displacement in the x direction; υ is the displacement in the y direction; w is the displacement in the z direction.

For the case of absence of external forces, the equations of motion can be expressed using the following relationships

$$\sigma_{xx} + \tau_{xy,y} + \tau_{xz,z} = \rho \ddot{u}, \tag{2}$$

$$\tau_{xy,x} + \sigma_{yy} + \tau_{yz,z} = \rho \ddot{\upsilon}, \tag{3}$$

$$\tau_{xz,x} + \tau_{yz,y} + \sigma_{zz} = \rho \ddot{w}, \tag{4}$$

where ρ is the density of fixed lamina.

The boundary conditions on the upper and lower surfaces can be written using the following equations

$$\sigma_z = \tau_{xz} = \tau_{yz} = 0, \qquad z = \pm \frac{h}{2}. \tag{5}$$

Lamb wave packets propagate along the plane of a plate with boundaries free of additional mechanical stresses. On the other hand, Lamb waves are standing waves in the *z*-direction of the plate. Therefore, the wave motion can be expressed by a superposition of plane harmonic waves. Each plane harmonic wave propagating in the direction of the wave normal *k* can be described by the relation

$$\{u, \upsilon, w\} = \{U(z), V(z), W(z)\} \exp\left\{ i \left[\left(k_x x + k_y y - \omega t \right) \right] \right\}, \tag{6}$$

where $k = [k_x, k_y]^T$.

Magnitude of *k* is

$$|k| = \sqrt{k_x^2 + k_y^2} = \frac{\omega}{c_p}, \tag{7}$$

$$k = \frac{2\pi}{\lambda}, \tag{8}$$

where λ is the wavelength; ω is the angular frequency; c_p is the phase velocity.

Mechanical stresses in each layer are

$$\sigma_x = \left[C_{11} k_x U + C_{12} k_y - i C_{13} W' + C_{16} \left(k_y U + k_x V \right) \right] \times$$
$$\times \exp\left\{ i \left[\left(k_x x + k_y y \right) - \omega t \right] \right\}, \tag{9}$$

$$\sigma_y = \left[C_{12} k_x U + C_{22} k_y - i C_{23} W' + C_{36} \left(k_y U + k_x V \right) \right] \times$$
$$\times \exp\left\{ i \left[\left(k_x x + k_y y \right) - \omega t \right] \right\}, \tag{10}$$

$$\sigma_z = \left[C_{31} k_x U + C_{32} k_y - i C_{33} W' + C_{63} \left(k_y U + k_x V \right) \right] \times$$

$$\times \exp\{i[(k_x x + k_y y) - \omega t]\} , \quad (11)$$

$$\tau_{yz} = [C_{44}(V' + ik_y W) + C_{45}(U' + ik_x W)] \times$$
$$\times \exp\{i[(k_x x + k_y y) - \omega t]\} , \quad (12)$$

$$\tau_{xz} = [C_{54}(V' + ik_y W) + C_{55}(U' + ik_x W)] \times$$
$$\times \exp\{i[(k_x x + k_y y) - \omega t]\} , \quad (13)$$

$$\tau_{xy} = [C_{61}k_x U + C_{62}k_y V - iC_{63}W' + C_{66}(k_y U + k_x V)] \times$$
$$\times \exp\{i[(k_x x + k_y y) - \omega t]\} , \quad (14)$$

The equations for mechanical displacements for an off-axis composite plate allow separation into symmetric (index "s") and antisymmetric (index "a") wave modes. This separation leads to a particularly simple form of the asymptotic representation

$$U_s = A_s \cos \xi z , \quad V_s = B_s \cos \xi z , \quad W_s = C_s \sin \xi z , \quad (15)$$

$$U_a = A_a \cos \xi z , \quad V_a = B_a \cos \xi z , \quad W_a = C_a \sin \xi z , \quad (16)$$

where ξ is the fixed variable.

The method of analyzing the resulting system of equations for mechanical displacements and stresses can be divided into two successive stages. In the first approximation, only symmetrical modes of Lamb waves are subjected to theoretical analysis during their group motion along the anisotropic medium, which constitutes the volume of the laminated composite. In addition, a set of compact dispersion relations is analyzed separately for both symmetric and antisymmetric Lamb wave modes. This analysis is performed using metric functions for the corresponding wavelet transform for all points in the laminated composite volume. Symmetrical modes are considered first. At the second stage, the entire sequence of solutions of this system is transformed into a matrix form

$$\begin{bmatrix} \Gamma_{11} - \rho\omega^2 & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} - \rho\omega^2 & \Gamma_{23} \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} - \rho\omega^2 \end{bmatrix} \begin{Bmatrix} A_s \\ B_s \\ C_s \end{Bmatrix} = 0 , \quad (17)$$

Matrix elements can be expressed by the following expressions

$$\Gamma_{11} = C_{11}k_x^2 + 2C_{61}k_x k_y + C_{66}k_y^2 + C_{55}\xi^2, \tag{18}$$

$$\Gamma_{12} = C_{61}k_x^2 + (C_{12} + C_{66})k_x k_y + C_{62}k_y^2 + C_{45}\xi^2, \tag{19}$$

$$\Gamma_{13} = -i[(C_{31} + C_{55})k_x + (C_{63} + C_{45})k_y]\xi \ , \tag{20}$$

$$\Gamma_{22} = C_{66}k_x^2 + 2C_{26}k_x k_y + C_{22}k_y^2 + C_{44}\xi^2, \tag{21}$$

$$\Gamma_{23} = -i[(C_{36} + C_{45})k_x + (C_{23} + C_{44})k_y]\xi \ , \tag{22}$$

$$\Gamma_{33} = C_{55}k_x^2 + 2C_{45}k_x k_y + C_{44}k_y^2 + C_{33}\xi^2, \tag{23}$$

A similar technique is consistently applied to the antisymmetric mode. As a result, the resulting matrix becomes equal to $(\mathbf{\Gamma} - \rho\omega^2\mathbf{I})$, where $\mathbf{I}$ is the identity matrix. For the case when the Hermitian matrix $\mathbf{\Gamma}$ is positive definite, it can be shown that the eigenvalues of the symmetric and antisymmetric modes coincide.

The non-trivial solutions A_s, B_s and C_s participate in the zeroing of the determinant of the characteristic matrix $(\mathbf{\Gamma} - \rho\omega^2\mathbf{I})$, which yields the following sixth-order polynomial in ξ

$$\xi^6 + \alpha_1\xi^4 + \alpha_2\xi^2 + \alpha_3 = 0, \tag{24}$$

where α_i are real-valued coefficients of C_{ij}, k, and $\rho\omega^2$.

Analysis of the system of characteristic equations showed that there are three solutions. The properties of these solutions include their positivity, nonzero difference, and discreteness. A_s is a rule, such solutions are related to indices n_j (j = 1, 2, 3). For each index nj in the symmetric modes of the wave packets B_s and C_s, which are related to the symmetric modes of Lamb waves in composite structures, the fixed relationships can be written in terms of A_s

$$B_s = \frac{(\Gamma_{11} - \rho\omega^2)\Gamma_{23} - \Gamma_{12}\Gamma_{13}}{\Gamma_{13}(\Gamma_{22} - \rho\omega^2) - \Gamma_{12}\Gamma_{23}} A_s = RA_s, \tag{25}$$

$$C_s = \frac{\Gamma_{12}^2 - (\Gamma_{11} - \rho\omega^2)(\Gamma_{22} - \rho\omega^2)}{\Gamma_{13}(\Gamma_{22} - \rho\omega^2) - \Gamma_{12}\Gamma_{23}} A_s = iSA_s. \tag{26}$$

As a result, the modified equations for mechanical shears and stresses will have the form

$$\left(\sigma_z, \tau_{yz}, \tau_{xz}\right)\Big|_{z=h/2} = \sum_{j=1}^{3}[H_{1j} \sin\left(\xi_j z + \varphi\right), H_{2j} \cos\left(\xi_j z + \varphi\right) +$$

$$+ H_{3j} \cos\left(\xi_j z + \varphi\right)]A_j = 0, \qquad (27)$$

where φ = 0 and π/2 represent anti-symmetric and symmetric Lamb wave modes.

The numerical calculation methodology for Lamb wave propagation in composite structures assumes that the interfaces between layers are ideally coupled. For each layer, the displacement components in the corresponding z-axis equation must be modified into exponential forms to account for the inhomogeneity of the multilayer laminate.

Symmetrical and asymmetrical wave modes in conventional laminates cannot be separated. It should be noted that symmetrical laminates are used in engineering practice when designing composite structures. A reliable method for separating the two types of wave modes is to generate boundary conditions on both the upper and middle planes of the surface.

Summary and conclusions.

The results of the analysis indicate that the propagation of plane elastic waves in non-isotropic laminar composites can be conveniently described using wavelet transforms, which allow us to reduce this problem to the propagation of symmetric and antisymmetric Lamb waves in local volumes. The direction of propagation of Lamb wave packets coincides with the direction along the plane of the plate with boundaries free of additional mechanical stresses. Calculations show the convenience of representing Lamb wave packets as standing waves. Therefore, wave motion can be expressed by a superposition of plane harmonic waves. Each plane harmonic wave propagates in the direction of the wave normal. The results of the work can be considered as an optimization of the method of flaw detection of laminar composites using Lamb waves.

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