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APPROXIMATION METHOD FOR A MULTILAYER TRANSVERSELY ISOTROPIC MATERIAL

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Abstract. *This study analyzes in detail the advantages of using an approximation method to describe the propagation of Lamb wave packets in multilayer structures using laminated composites as an example. It is shown that the approximation technique for a multilayer transversely isotropic material becomes most effective with a preliminary detailed analysis of the set of stiffness characteristics for each of the composite layers separately. This analysis assumes the possibility of approximation by polynomial interpolation functions for the displacement distribution throughout the volume of the laminated structure. In addition, as an addition, a hybrid method was considered, which assumes the use of a semi-analytical finite element. The finite element method was used to discretize the cross-section and described the displacement along the Lamb wave propagation using analytical simple harmonic functions. It was found that the most accurate method for calculating the propagation characteristics of Lamb waves in composites is the method of linear 3D elastic properties. A correct description of the propagation of Lamb wave packets is possible only for the case of orthotropic composite materials.*

Key words: *multilayer material, Lamb waves, polynomial interpolation functions, finite element method, orthotropic composites.*

Introduction.

Lamb waves are a type of guided elastic wave that propagate in solid plates with free boundaries. Due to their ability to travel long distances with relatively low attenuation, they are widely used in nondestructive evaluation (NDE) of materials, particularly thin-walled structures such as composite laminates [1]. In composite materials, the anisotropy and layered structure significantly affect Lamb wave propagation. The wave velocity, mode shapes, and attenuation depend on both the elastic properties and the geometry of the laminate. Lamb waves exist in two fundamental modes: symmetric (S) and antisymmetric (A) [2]. These modes further split into higher-order modes at increased frequencies. The S0 and A0 modes are the most commonly studied for low-frequency applications.

The dispersion behavior of Lamb waves is critical for their application in composites. Dispersion curves, which relate frequency and phase/group velocity, are necessary for understanding how different modes behave in complex media.

Composite laminates introduce complexity in Lamb wave analysis due to their heterogeneous and direction-dependent properties. The stiffness matrix varies with fiber orientation and stacking sequence, making analytical solutions more challenging than for isotropic materials. Numerical methods, such as the finite element method (FEM) and the semi-analytical finite element (SAFE) method, are widely used to model Lamb wave propagation in composite plates [3, 4]. These models help in predicting wave modes, velocities, and their sensitivity to defects. Experimental techniques such as laser Doppler vibrometry, scanning acoustic microscopy, and piezoelectric transducers are commonly used to excite and measure Lamb waves in composites. The selection of the excitation frequency and sensor placement significantly influences the detectability of different modes.

In composites, mode conversion is frequently observed due to the interaction of Lamb waves with discontinuities, such as delaminations or material interfaces. This phenomenon can provide valuable information about the type and location of defects. The presence of defects such as delaminations, voids, and fiber breakage alters the Lamb wave propagation characteristics. These changes manifest as variations in amplitude, phase, or arrival time, which can be used for damage detection. Wavelet transform techniques are often employed to analyze the nonstationary signals generated by Lamb waves in composites. These methods enhance the ability to extract time-frequency features related to damage.

Dispersion compensation algorithms are sometimes applied to improve signal clarity and facilitate the identification of specific Lamb wave modes. This is particularly useful when working with broadband excitation signals. The directionality of Lamb wave propagation in anisotropic media affects the energy distribution and sensitivity of inspection methods. Understanding this behavior is crucial for optimizing sensor layouts in structural health monitoring (SHM) systems.

Temperature effects can influence Lamb wave velocities and mode shapes. Therefore, compensation or calibration is often necessary when applying Lamb wave-based techniques in varying environmental conditions. Recent advancements in machine learning and signal processing are being integrated with Lamb wave analysis to improve defect classification and localization in composite structures. Overall, the propagation of Lamb waves in composites is a complex but powerful tool for NDE and SHM. By accounting for material anisotropy, mode behavior, and wave interactions, engineers can develop effective diagnostic systems for maintaining the integrity of composite structures.

One of the simplest methods for generating Lamb wave dispersion curves is to use the effective stiffness approach. In this methodology, a geometrically weighted average of the component property values is used as the average material constants for the entire laminate. An additional methodology is based on the classical laminated plate theory. It should be noted that the classical laminated plate theory cannot accurately predict the dispersion behavior of Lamb waves at sufficiently high frequencies.

Both analytical techniques have high computational efficiency. However, the classical laminated plate theory and higher order theories are only approximations and cannot accurately predict the higher modes of Lamb waves at higher frequencies. The solution to this problem, at least partially, is the approximation method for a multilayer transversely isotropic material. The approximation method is based on the analysis of a set of stiffness characteristics. Such an analysis assumes the possibility of approximation by polynomial interpolation functions for the thickness displacement distribution.

An additional hybrid method involves the use of a semi-analytical finite element, which uses the finite element method to discretize the cross-section and describes the displacement along the wave propagation using analytical simple harmonic functions. The most accurate method for calculating the propagation characteristics of Lamb waves in composites is the linear 3D elastic properties method.

Approximation method for a multilayer material.

The computational method uses the partial wave method in combination with the global matrix approach to numerically solve the Lamb wave equations. A robust step-by-step solution for generating Lamb wave dispersion curves is the main objective for this part of the Lamb wave packet propagation analysis.

First of all, it is necessary to consider the shear-stress relationship. The tensor form of the stress-strain relationship in a Cartesian coordinate system for an anisotropic solid medium assuming linear elastic behavior is as follows

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl}\,, \qquad (1)$$

$$\varepsilon_{ij} = s_{ijkl}\sigma_{kl}\,, \qquad (2)$$

where

c_{ijkl} is the stiffness tensor;

s_{ijkl} is the compliance tensor.

The dependence of the linear elastic deformation on the mechanical displacement can be determined by the following relationship

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \qquad (3)$$

In turn, the generalized equation of motion is determined by the components of the displacement

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}\,. \qquad (4)$$

Each layer in the laminar composite according to the calculation model is described by a local $\{x_1*, x_2*, x_3*\}$ and global $\{x_1, x_2, x_3\}$ coordinate system.

The mechanical stress in the global system is equal to

$$\{\sigma\} = [T_\sigma]\{\sigma *\}, \qquad (5)$$

where $[T_\sigma]$ is the stress transformation matrix.

The transformation of the stiffness matrix from the local to the global coordinate system can be performed using the following algorithm

$$[c] = [T_\sigma][c*][T_\varepsilon]^{-1}. \qquad (6)$$

The propagation of Lamb wave packets is described by the governing equations for the case of composite materials that exhibit orthotropic and higher degrees of symmetry.

It should be noted, however, that it is necessary to consider lower monoclinic symmetry for the excitation and propagation mode of wave packets in an orthotropic or transversely isotropic laminate along a non-principal direction, or if the stacking is symmetric but not balanced. For example, this will be observed as a result of the installation of wave signal generators that can be fixed in a non-principal direction of the commonly used orthotropic or lower symmetry of the plate.

These factors lead to the fact that the Lamb wave equations will be derived for monoclinic symmetry, which can be used for any material symmetry that is higher than monoclinic.

The stress-strain relationship for monoclinic composite material can be expressed as

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & c_{16} \\ c_{12} & c_{22} & c_{23} & 0 & 0 & c_{26} \\ c_{13} & c_{23} & c_{33} & 0 & 0 & c_{36} \\ 0 & 0 & 0 & c_{44} & c_{45} & 0 \\ 0 & 0 & 0 & c_{45} & c_{55} & 0 \\ c_{16} & c_{26} & c_{36} & 0 & 0 & c_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{13} \\ 2\varepsilon_{12} \end{Bmatrix}. \quad (7)$$

Let us consider in more detail the model of Lamb wave propagation in a monoclinic material in order to derive representative equations. At the first stage, we will analyze the displacement field in all three directions in comparison with the consideration of only the propagation of wave packets along the principal directions.

The consideration of Lamb waves can be non-isotropic for all directions in a laminar composite sample. Substituting the displacement fields U_j with the general equilibrium equations for the displacement field have the form

$$K_{ij}(k_3)U_j = 0, \quad (8)$$

where the matrix elements have the form

$$K_{11} = c_{11}k_1^2 + c_{66}k_2^2 + c_{55}k_3^2 + 2c_{16}k_1k_2 - \rho\omega^2, \quad (9)$$

$$K_{12} = c_{16}k_1^2 + c_{26}k_2^2 + c_{45}k_3^2 + (c_{12} + c_{66})k_1k_2, \quad (10)$$

$$K_{13} = (c_{13} + c_{55})k_1k_3 + (c_{36} + c_{45})k_2k_3, \quad (11)$$

$$K_{22} = c_{66}k_1^2 + c_{22}k_2^2 + c_{44}k_3^2 + 2c_{26}k_1k_2 - \rho\omega^2, \quad (12)$$

$$K_{23} = (c_{36} + c_{45})k_1k_3 + (c_{23} + c_{44})k_2k_3, \quad (13)$$

$$K_{33} = c_{55}k_1^2 + c_{44}k_2^2 + c_{33}k_3^2 + 2c_{45}k_1k_2 - \rho\omega^2. \quad (14)$$

The condition for the presence of a set of non-trivial solutions to a system of equations can be reduced to the form

$$\det(K_{ij}) = 0 \quad (15)$$

or

$$D_1k_3^6 + D_2k_{31}^4 + D_3k_3^2 + D_4 = 0. \quad (16)$$

The three roots of k_3^2 correspond to one pair of quasi-longitudinal and two pairs of quasi-shear modes. The six roots of k_3 can be divided into three pairs, with the constituent elements of each pair being negative with respect to each other. Each pair represents an ascending and descending traveling wave making the same angle with the x_1 axis.

The relationship between mechanical stresses and stiffening elements for the case of boundary conditions without tension takes the form

$$\sigma_{33} = c_{13}\frac{\partial u_1}{\partial x_1} + c_{23}\frac{\partial u_2}{\partial x_2} + c_{33}\frac{\partial u_3}{\partial x_3} + c_{36}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right), \quad (17)$$

$$\sigma_{13} = c_{45}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) + c_{55}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right), \quad (18)$$

$$\sigma_{23} = c_{44}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) + c_{45}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right). \quad (19)$$

Free boundary conditions at the top and bottom surfaces of lamina composites are

$$\sigma_{33}\big|_{x_3=\pm h} = \sigma_{13}\big|_{x_3=\pm h} = \sigma_{23}\big|_{x_3=\pm h} = 0. \quad (20)$$

In turn, displacements can be expressed as a function of amplitude.

Summary and conclusions.

The dependence of the linear elastic deformation on the mechanical displacement is not of a very complex polynomial nature. However, the correct recording of the generalized equation of motion requires the most complete data on all components of the displacement in each layer. A limitation of the approximation method can be